Three plus one equilateral triangles.

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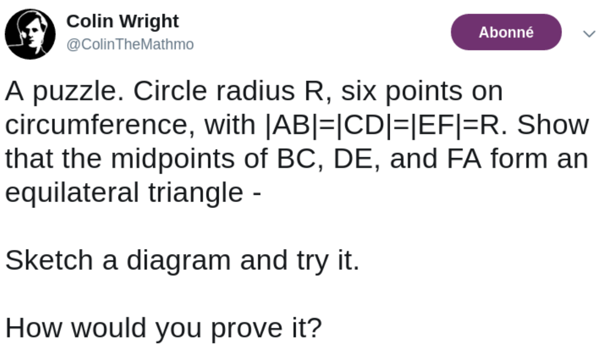
I propose a solution to the following problem posted by Colin Wright on Twitter :


Colin Wright
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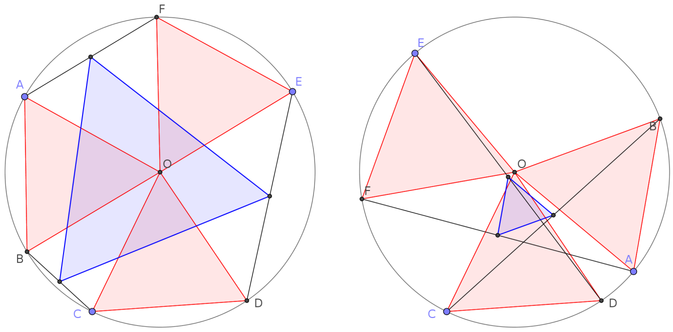
A puzzle. Circle radius R, six points on circumference, with |AB|=|CD|=|EF|=R. Show that the midpoints of BC, DE, and FA form an equilateral triangle -

Sketch a diagram and try it.

How would you prove it?



Proof. Without any loss of generality we can suppose that the points are on the unit circle centered at the origin. Let us write $\omega = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$. We have:

$$z_B = \omega z_A, \quad z_D = \omega z_C, \quad z_F = \omega z_E$$

We want to prove that the midpoints of $[AF]$, $[BC]$ and $[DE]$ form an equilateral triangle, *i.e.* :

$$|z_A + z_F - (z_E + z_D)| = |z_B + z_C - (z_E + z_D)| = |z_B + z_C - (z_A + z_F)|$$

Which is equivalent to :

$$|z_A + \omega z_E - (z_E + \omega z_C)| = |\omega z_A + z_C - (z_E + \omega z_C)| = |\omega z_A + z_C - (z_A + \omega z_E)|$$

Using $\omega - 1 = \omega^2$ this is equivalent to :

$$|z_A - \omega z_C + \omega^2 z_E| = |\omega z_A - \omega^2 z_C - z_E| = |\omega^2 z_A + z_C - \omega z_E|$$

Or equivalently, using $\omega^3 = -1$ (the famous $\mathrm{e}^{\mathrm{i}\pi} = -1$!) :

$$|z_A - \omega z_C + \omega^2 z_E| = |\omega z_A - \omega^2 z_C + \omega^3 z_E| = |\omega^2 z_A - \omega^3 z_C + \omega^4 z_E|$$

If you set $Z = z_A - \omega z_C + \omega^2 z_E$ this equation can be written as : $|Z| = |\omega Z| = |\omega^2 Z|$ and it is true because $|\omega| = |\omega^2| = 1$. □